

Heat-kernel Coefficients and Spectra of the Vector Laplacians on Spherical Domains with Conical Singularities

Lara De Nardo¹, Dmitri V. Fursaev^{2,3} and Gennaro Miele¹

¹ *Dipartimento di Scienze Fisiche, Università di Napoli - Federico II -, and INFN Sezione di Napoli, Mostra D'Oltremare Pad. 20, 80125, Napoli, Italy*

² *Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1*

³ *Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, Dubna Moscow Region, Russia*

Abstract

The spherical domains S^d_β with conical singularities are a convenient arena for studying the properties of tensor Laplacians on arbitrary manifolds with such a kind of singular points. In this paper the vector Laplacian on S^d_β is considered and its spectrum is calculated exactly for any dimension d . This enables one to find the Schwinger-DeWitt coefficients of this operator by using the residues of the ζ -function. In particular, the second coefficient, defining the conformal anomaly, is explicitly calculated on S^d_β and its generalization to arbitrary manifolds is found. As an application of this result, the standard renormalization of the one-loop effective action of gauge fields is demonstrated to be sufficient to remove the ultraviolet divergences up to the first order in the conical deficit angle.

PACS number(s): 04.60.+n, 12.25.+e, 97.60.Lf, 11.10.Gh

1 Motivations and results

Quantum field theory on spaces \mathcal{M}_β with conical singularities is a research subject which recently many publications have been devoted to. Conical singularities are a set Σ of points near which \mathcal{M}_β have the structure $\mathcal{C}_\beta \times \Sigma$ where \mathcal{C}_β is a cone with an angle β . The well-known example of such situation in physics is a space-time around an idealized infinitely thin cosmic string [1].

Spaces like \mathcal{M}_β appear in the black hole thermodynamics as well, and for this reason they have been carefully investigated in the last years. In the Gibbons–Hawking formulation [2] the free energy of a black hole is defined in terms of the gravitational action on the Euclidean black hole instanton. The regularity condition of such instantons near the Euclidean horizon picks up a period β of the imaginary time which is equal to the inverse Hawking temperature T_H^{-1} . On the other hand, the mass M of a black hole is uniquely related to its Hawking temperature T_H and, hence the Hamiltonian describing quantum excitations on the black hole background depends on the temperature as well [3]. However, the statistical–mechanical definition of the entropy and other quantities usually requires calculation of the partial derivatives of the free energy on the temperature. For the black holes it implies that one has to deal with the temperatures which are different from the Hawking value and independent on the mass M . In this case the black hole instanton is a space like \mathcal{M}_β , and has a conical singularity on the Euclidean horizon Σ . The corresponding approach to the black hole thermodynamics is called the off-shell approach [4, 5], because singular spaces do not obey the vacuum Einstein equations. In the Einstein gravity the off-shell derivation gives the correct Bekenstein–Hawking value for the black hole entropy [6]. The off-shell method can be also generalized to the higher-derivative gravity theories [7] and it enables one to get the black hole entropy which agrees with the other methods.

The search for the statistical–mechanical explanation of the black hole entropy [8] has given rise to the off-shell methods in the thermodynamics of quantum black holes (see for review [5]). The methods, which use manifolds with conical singularities, were formulated and investigated for the static [10]–[12],[5] and for the rotating black holes [13, 14]. On curved backgrounds the one-loop effective action W is defined in terms of the trace $K(s)$ of the heat kernel for the Laplace operator Δ of the corresponding field, by equation

$$W = -\frac{1}{2} \log \det \Delta = -\frac{1}{2} \int_{\delta^2}^{\infty} \frac{ds}{s} K(s) \quad , \quad (1.1)$$

where δ^2 is an ultraviolet cut-off. The important fact is that the conical singularities change the structure of the ultraviolet divergences W_{div} of the action (1.1), which in four dimensions has the form

$$W_{\text{div}} = -\frac{1}{32\pi^2} \left(\frac{1}{2\delta^4} A_0 + \frac{1}{\delta^2} A_1 - \ln \delta^2 A_2 \right) \quad , \quad (1.2)$$

because the coefficients A_n of the heat kernel expansion [15]

$$K(s) = \frac{1}{(4\pi s)^2} (A_0 + s A_1 + s^2 A_2 + \dots) \quad (1.3)$$

acquire for $n \geq 1$ the additional contributions in the form of integrals on the singular surface Σ [16], [17]–[21]. As a consequence of this, the off-shell entropy of the quantum fields calculated on \mathcal{M}_β has the divergence proportional to the area of the horizon Σ . This phenomenon has been investigated in detail for scalar fields. In particular, by following the idea of Ref.[8], the important property that the divergencies of the off-shell black hole entropy are completely removed under standard renormalization [22] of the Newton constant and other couplings in the bare gravitational action was proven [23].

The analogous properties of the effective actions of higher spin fields are not well-understood so far. For the vector and Dirac fields the change of the coefficient A_1 in (1.3) because of conical singularities has been found in [24, 25] and the corresponding ultraviolet divergence is shown to be removed under renormalization of the Newton constant [24]–[26], as in scalar case. However, as follows from the analysis of [25], the heat kernel expansion for the spin 3/2 Laplacian and for the Lichnerowicz operator on \mathcal{M}_β has a feature which makes them different from the case of the scalar operator.

This paper studies the vector Laplacians on singular spaces \mathcal{M}_β and their implications in quantum theory. The way we have chosen is to consider these operators on d -dimensional spherical domains, denoted further as S_β^d , and find their spectrum exactly. Note that in general finding these spectra on curved manifolds is a quite difficult task, and backgrounds where it can be done exactly are of a particular interest.

The space S_β^d can be described by the metric

$$ds^2 = a^2 (\cos^2 \chi d\tau^2 + d\chi^2 + \sin^2 \chi dl^2) \quad , \quad (1.4)$$

where $0 \leq \tau \leq \beta$, $|\chi| \leq \pi/2$ and dl^2 is the line element on the $(d-2)$ -dimensional unit sphere. The points with coordinates $\chi \simeq \pm\pi/2$ form the sphere S^{d-2} near which S_β^d has the structure $\mathcal{C}_\beta \times S^{d-2}$. Outside this domain S_β^d coincides with the d -dimensional hypersphere S^d with radius a . It should be noted that spaces S_β^d naturally appear in studying the finite-temperature quantum field theory in static de Sitter space [27].

We show that the Laplace operator $-\nabla_\mu \nabla^\mu$ for the transverse vector field on S_β^d has the following spectrum

$$\Lambda_{n,m}^T(d) = \frac{1}{a^2} [(n + \gamma m)(n + \gamma m + d - 1) - 1] \quad , \quad (1.5)$$

where $\gamma = 2\pi/\beta$, $n = 0, 1, 2, \dots$ when $m = 1, \dots$, and $n = 1, 2, \dots$ when $m = 0$. The eigen-values with $m = 0$ have the degeneracy

$$D_{n,0}^T(d) = \frac{1}{n+1} \binom{d+n-3}{n-1} [d^2 + (n-4)d + (5-n)] \quad , \quad (1.6)$$

while for $m \neq 0$ one has

$$D_{n,m}^T(d) = 2(d-1) \binom{d+n-2}{n} . \quad (1.7)$$

Then, by using the properties of the ζ -function on S_β^d and an assumption about the structure of heat kernel coefficient $A_2^{(1)}$ for the vector Hodge-deRham operator $\Delta^{(1)}$ we find explicitly the addition

$$\begin{aligned} A_{\beta,2}^{(1)} = \frac{\pi}{3\gamma} \int_\Sigma \left[(1-\gamma^4) \frac{d}{30} \left(\frac{1}{2} R_{ii} - R_{ijij} \right) - (1-\gamma^2) \frac{d}{6} R \right. \\ \left. + (1-\gamma^2)(R - R_{ii}) + 2(1-\gamma)(R - 2R_{ii} + R_{ijij}) \right] \end{aligned} \quad (1.8)$$

to this coefficient on the arbitrary spaces \mathcal{M}_β with the conical singularities on Σ . Here, R is the scalar curvature and R_{ii} , R_{ijij} are invariants constructed of the components of the Riemann and Ricci tensors normal to Σ and computed near this surface. The knowledge of Eq.(1.8) enables one to make the conclusion that the heat-kernel coefficients of the vector Hodge-deRham operator have the same properties as those of the scalar Laplacians. In particular, the theorem of Ref.[23] about the renormalization of the entropy, calculated in the conical singularities method for scalar fields, also holds for vector fields.

The paper is organized as follows. In Section 2 we start by considering the scalar Laplacian, and use this case to describe our strategy and derive some basic formulas. Section 3 is devoted to the spectrum of the vector Laplacian. The first $A_1^{(1)}$ and second $A_2^{(1)}$ coefficients in the heat kernel expansion of the vector Hodge-deRham operator on S_β^d are found in Section 4. We use to this aim the relation between the residues of the ζ -function and the Schwinger-DeWitt coefficients. The renormalization of the effective action of the vector fields on \mathcal{M}_β is discussed in Section 5 after that the results are summarized in Section 6. Some useful formula and technical details can be found in the Appendices A and B.

2 The method: spectrum of the scalar Laplacian

To find the spectra we will follow the method used in the literature for the Laplace operators on the hyperspheres S^d (see for instance [28]). A d -sphere S^d can be embedded into the Euclidean space \mathbb{R}^{d+1} . Let x^K , $K = 1, \dots, d+1$, be the Cartesian coordinates in \mathbb{R}^{d+1} and $(r, \tau, \chi, \theta^i) = (r, u^\mu)$, $\mu = 1, \dots, d$, be the spherical ones. The relation between the two coordinate frames reads

$$x^{d+1} = r \cos \tau \cos \chi, \quad x^d = r \sin \tau \cos \chi, \quad x^k = r \sin^k(\theta) \sin \chi, \quad k = 1, \dots, d-1, \quad (2.1)$$

where θ^i ($i=1, \dots, d-2$) parameterize the unit hypersphere $\sum_k n_k^2 = 1$. If the embedding of S^d of radius a in \mathbb{R}^{d+1} is described by equation $x_K x^K = a^2$, then the line element on S^d has the form (1.4).

In spherical coordinates the vector field which is in the tangent space to S^d has components $V^r = 0, V^\mu$. For such a field one has the following relation [28, 29]

$$-\nabla^\mu \nabla_\mu V_\nu = \left[-\nabla^K \nabla_K + \left(\frac{\partial}{\partial r} + \frac{d-1}{r} \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) - \frac{1}{r^2} \right] V_\nu \quad (2.2)$$

between the Laplacian $-\nabla^\mu \nabla_\mu$ on the hypersphere S^d with the radius r , and the Laplacian $-\nabla^K \nabla_K$ in \mathbb{R}^{d+1} . Therefore, by making use of (2.2) one can reduce the problem of finding the spectrum for the vector Laplacian on the hypersphere to the eigen-problem for the operator $-\nabla^K \nabla_K$ in the flat space, where the latter has a much more easy solution.

To show that the same method is applicable for the Laplace operator on the singular d -spheres S_β^d let us consider the locally flat space \mathbb{R}_β^{d+1} . If $\beta < 2\pi$, then \mathbb{R}_β^{d+1} can be obtained from \mathbb{R}^{d+1} under identification of the points with the coordinates $x^K(\tau)$ and $x^K(\tau + \beta)$. When $\beta > 2\pi$, one has at first to replace \mathbb{R}^{d+1} with the space \mathbb{R}_∞^{d+1} , which has an infinite range for the polar coordinate τ defined in (2.1), and represents an analog of the Riemann sheet. Then \mathbb{R}_β^{d+1} appears after the identification of the points $x^K(\tau)$ with $x^K(\tau + \beta)$ of \mathbb{R}_∞^{d+1} .

Although \mathbb{R}_β^{d+1} is a locally flat space, it has the conical singularities on the $(d-1)$ -dimensional hyperplane $x^{d+1} = x^d = 0$ ($\chi = \pm\pi/2$). The singular sphere S_β^d with the metric (1.4) can be embedded into \mathbb{R}_β^{d+1} . This embedding is described by the same equation $x_K x^K = a^2$ as in the previous case, provided the relation between Cartesian coordinates x^K and spherical ones $(r, \tau, \chi, \theta^i)$ is left unchanged. As it is easy to understand, the relation between the vector Laplacian in \mathbb{R}_β^{d+1} and on S_β^d does not change and coincides with the Eq.(2.2). Therefore for the spectral problem it is sufficient to investigate the operator $-\nabla^K \nabla_K$ in this locally flat space.

At first it is worth applying the above technique to find the spectrum for the scalar Laplacian on S_β^d . The results will be useful then in studying the vector case. Let us consider to this aim the scalar field on S_β^d , whose wave functions ϕ belong to the Hilbert space $L^2(S_\beta^d)$ and are periodic when τ is increased by β

$$\phi(\tau + \beta, u) = \phi(\tau, u) \quad . \quad (2.3)$$

We show below that for $\beta \leq 2\pi$ the eigen-functions $\phi_{n,m}^\pm$ of this operator can be represented as homogeneous polynomials in the Cartesian coordinates

$$\phi_{n,m}^\pm = r^{-(n+\gamma m)} \sum_{p=0}^{[n/2]} C_{i_1 \dots i_{n-2p}}^{\pm m} x^{i_1} \dots x^{i_{n-2p}} (x^+ x^-)^p (x^\pm)^{\gamma m} \quad , \quad (2.4)$$

where $[n/2]$ is the integer part of $n/2$, and $C_{i_1 \dots i_{n-2p}}^{\pm m}$ are symmetric tensors. Note that the coordinates x^\pm , defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}} (x^{d+1} \pm i x^d) \quad , \quad (2.5)$$

transform as $x^\pm(\tau + \beta) = e^{\pm i\beta} x^\pm(\tau)$. The corresponding eigen-values for both $\phi_{n,m}^+$ and $\phi_{n,m}^-$ read

$$\Lambda_{n,m}^{(0)} = \frac{1}{a^2} (n + \gamma m)(n + \gamma m + d - 1) \quad , \quad n, m = 0, 1, 2, \dots \quad (2.6)$$

and have the following degeneracy

$$D_{n,m=0}^{(0)}(d) = \binom{n + d - 2}{n} \quad (2.7)$$

and $D_{n,m \neq 0}^{(0)}(d) = 2D_{n,m=0}^{(0)}(d)$.

To prove this result let us consider a polynomial of an arbitrary form

$$\phi_{n,\omega,\rho} = r^{-(n+\omega+\rho)} \sum_{q=0}^n C_{i_1 \dots i_{n-q}}^{\omega,\rho} x^{i_1} \dots x^{i_{n-q}} (x^+ x^-)^{\frac{1}{2}q} (x^+)^{\omega} (x^-)^{\rho} \quad , \quad (2.8)$$

where ω and ρ are some constants which do not depend on n . This polynomial obeys the generalized boundary condition

$$\phi_{n,\omega,\rho}(\tau + \beta, u) = e^{i(\omega-\rho)\beta} \phi_{n,\omega,\rho}(\tau, u) \quad , \quad (2.9)$$

and does not depend on r ,

$$\frac{\partial}{\partial r} \phi_{n,\omega,\rho} = 0 \quad . \quad (2.10)$$

As it is easy to see the functions (2.8) belong to $L^2(S_\beta^d)$ if

$$\rho + \omega > -1 \quad . \quad (2.11)$$

A direct computation shows that $\phi_{n,\omega,\rho}$ are eigen-functions of the operator $-\nabla^K \nabla_K$

$$-\nabla^K \nabla_K \phi_{n,\omega,\rho} = \frac{1}{r^2} (n + \omega + \rho)(n + \omega + \rho + d - 1) \phi_{n,\omega,\rho} \quad (2.12)$$

provided the following conditions:

$$\omega \rho C_{i_1, \dots, i_n}^{\omega,\rho} = 0 \quad , \quad (2.13)$$

$$(2\omega + 1)(2\rho + 1) C_{i_1, \dots, i_{n-1}}^{\omega,\rho} = 0 \quad , \quad (2.14)$$

$$\frac{1}{2} (2\omega + q)(2\rho + q) C_{i_1, \dots, i_{n-q}}^{\omega,\rho} + (n - q + 2)(n - q + 1) C_{j,j,i_1, \dots, i_{n-q}}^{\omega,\rho} = 0 \quad (2.15)$$

are imposed. It is assumed in Eq.(2.15) that $q \geq 2$ and the summation is taken over the repeating indexes. If the coefficient $C_{i_1, \dots, i_n}^{\omega,\rho} \neq 0$, it defines all other coefficients $C_{i_1, \dots, i_{n-q}}^{\omega,\rho}$ with $q = 2p$. In this case equation (2.13) can be satisfied when $\omega = 0$ or $\rho = 0$. Then condition (2.14) holds when all other coefficients $C_{i_1, \dots, i_{n-q}}^{\omega,\rho}$ with odd $q = 2p + 1$ are zero.

For the field (2.3) the additional periodicity condition has to be imposed. Then, the form (2.4) of the eigen-vectors $\phi_{n,m}^\pm$ follows from Eq.(2.8) where the coefficients for odd q 's have to be omitted. The functions with $\omega = 0$ and $\rho = m\gamma$ correspond to the eigen-modes $\phi_{n,m}^-$, while the functions with $\rho = 0$ and $\omega = m\gamma$ correspond to $\phi_{n,m}^+$. If $\beta \leq 2\pi$ ($\gamma \geq 1$), then $m = 0, 1, 2, \dots$. If $\beta > 2\pi$, there may be a finite number of eigen-modes, belonging to $L^2(S_\beta^d)$, with negative m , which are, however, singular on the hypersurface $x^+ = x^- = 0$. In this case an additional analysis is required. For this reason our approach is to find the results for $\beta \leq 2\pi$ and then to extend them for any positive β by means of the analytical continuation.

Finally by taking into account Eqs.(2.10), (2.12) and the fact that the scalar operators $-\nabla^K \nabla_K$ and $-\nabla^\mu \nabla_\mu$ coincide, one gets the eigen-values (2.6) on the sphere S_β^d with radius $r = a$.

To compute the degeneracy, let us note that in (2.4) all the coefficients $C_{i_1, \dots, i_{n-2p}}^{\pm m}$ are determined by (2.15) in terms of the $(d-2)$ -dimensional rank- n symmetric tensor $C_{i_1, \dots, i_n}^{\pm m}$. The number of components

$$\binom{n+d-2}{n} \quad (2.16)$$

of $C_{i_1, \dots, i_n}^{\pm m}$ gives the number of independent eigen-modes $\phi_{n,m}^\pm$. The degeneracy $D_{n,m=0}^{(0)}$ coincides with the number of modes with $m = 0$. The degeneracy $D_{n,m \neq 0}^{(0)} = 2D_{n,m=0}^{(0)}$ is the total number of $\phi_{n,m}^+$ and $\phi_{n,m}^-$ because they correspond to the equal eigen-values.

It is instructive to see how the well-known spectrum of the Laplace operator on S^d follows from our results. If $\beta = 2\pi$ the eigen-values (2.6) depend on the number $l = n + m$ only

$$\Lambda_l^{(0)}(d) = \Lambda_{l,l-n}^{(0)}(d) = \frac{l(l+d-1)}{a^2} \quad , \quad \beta = 2\pi \quad . \quad (2.17)$$

The total degeneracy $D_l^{(0)}(d)$ corresponding to a given l is the sum

$$D_l^{(0)}(d) = D_{l,m=0}^{(0)}(d) + 2 \sum_{m=1}^l D_{l-m,0}^{(0)}(d) = \frac{(d+l-2)!}{l!(d-1)!} (2l+d-1) \quad , \quad (2.18)$$

which can be checked with the help of the relation

$$\sum_{k=0}^m \binom{h+k}{h} = \binom{h+m+1}{h+1} \quad . \quad (2.19)$$

The eigen-values (2.17) and degeneracies (2.18) coincide with the known results [28]. The spectrum of the scalar operator on S_β^d in the form (2.6), (2.7) was first established in [27] for the four-dimensional case.

3 Spectrum of the vector Laplacian

3.1 Eigen-values

We denote the Cartesian components of a vector with V^K . For a vector from the tangent space to S_β^d , whose components are defined with respect to the spherical coordinates, the periodicity condition takes the simple form

$$V^\mu(\tau + \beta) = V^\mu(\tau) \quad , \quad (3.1)$$

while in the Cartesian frame it looks as

$$V^L(\tau + \beta) = O_K^L(\beta) V^K(\tau) \quad . \quad (3.2)$$

The matrix $O_K^L(\beta) = \partial x^L(\tau + \beta) / \partial x^K(\tau)$ appears because the Cartesian basis $\partial / \partial x^L$ does not perform the complete rotation when τ is increased by β .

On d -spheres the eigen-value problem for the vector Laplacian is reduced to the separate problems for the transversal and longitudinal components of the vector field V_μ . The spectrum for the longitudinal part on S_β^d is completely determined by the scalar spectrum. This follows from the relation

$$- \left(\nabla^\mu \nabla_\mu \delta_\rho^\nu - R_\rho^\nu \right) \nabla_\nu \phi = - \nabla_\rho \left(\nabla^\mu \nabla_\mu \phi \right) \quad , \quad (3.3)$$

where ϕ is a scalar field and $R_\rho^\nu = a^{-2}(d-1)\delta_\rho^\nu$ is the Ricci tensor computed in the regular domain of S_β^d . For this reason we will consider further only transversal vectors.

At first let us find out the general form of the eigen-vectors V^L for the operator $-\nabla_K \nabla^K$ restricted to S_β^d . The results of Section 2 can be helpful for this aim, if we consider the basis connected with the coordinates (x^+, x^-, x^k) , $k = 1, \dots, d-1$. Then instead of the components V^d and V^{d+1} one has the components¹

$$V^\pm = \frac{1}{\sqrt{2}} \left(V^d \pm i V^{d+1} \right) \quad , \quad (3.4)$$

which transform under rotations (3.2) as

$$V^\pm(\tau + \beta) = e^{\pm i\beta} V^\pm(\tau) \quad . \quad (3.5)$$

Obviously, the components V^L can be treated just as a set of scalar fields. Thus, it follows from Eqs.(2.8),(2.12) and conditions (2.13),(2.14) that vector eigen-modes can be written in the following form

$$\begin{pmatrix} (V^+)_{n,m} \\ (V^k)_{n,m} \\ (V^-)_{n,m} \end{pmatrix} = r^{-(n+\gamma m)} \begin{pmatrix} (x^+)^{\gamma m+1} \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} (B^+)_{i_1, \dots, i_{n-2p-1}}^m x^{i_1} \dots x^{i_{n-2p-1}} (x^+ x^-)^p \\ (x^+)^{\gamma m} \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} (B^k)_{i_1, \dots, i_{n-2p}}^m x^{i_1} \dots x^{i_{n-2p}} (x^+ x^-)^p \\ (x^+)^{\gamma m-1} \sum_{p=0}^{\lfloor \frac{n+1}{2} \rfloor} (B^-)_{i_1, \dots, i_{n-2p+1}}^m x^{i_1} \dots x^{i_{n-2p+1}} (x^+ x^-)^p \end{pmatrix} \quad , \quad (3.6)$$

¹Note that $V_\mp = V^\pm$.

where the quantities $(V^+)_{0,m}$ are assumed to vanish. The range of n and m is $n = 0, 1, 2, \dots$ when $m = 1, 2, \dots$, and $n = 1, 2, \dots$ when $m = 0$. It is easy to check that for $n = m = 0$ there are no transverse modes. For simplicity we presented only one set of the eigen-modes. Another set can be obtained from (3.6) under interchange of x^+ with x^- . According to our notation, $(B^K)_{i_1, \dots, i_n}^m$ denote symmetric tensors, or constants, if the index is absent. We also impose the additional restriction at $m = 0$

$$(B^-)_{i_1, \dots, i_{n+1}}^0 = 0 \quad . \quad (3.7)$$

Then all the components in (3.6) are in the Hilbert space $L^2(S_\beta^d)$.

The vector (3.6) has the required periodicity and it is constructed in such a way that all its components have the same eigen-value for given numbers n and m . It can be verified with the help of (2.12) that

$$-\nabla^K \nabla_K (V^L)_{n,m} = \frac{1}{a^2} (n + \gamma m)(n + \gamma m + d - 1) (V^L)_{n,m} \quad , \quad (3.8)$$

provided the conditions

$$2p(p + \gamma m - 1)(B^-)_{i_1, \dots, i_{n-2p+1}}^m + (n - 2p + 2)(n - 2p + 3)(B^-)_{h, h, i_1, \dots, i_{n-2p+1}}^m = 0$$

$$\text{for } 1 \leq p \leq \left\lfloor \frac{n+1}{2} \right\rfloor \quad , \quad (3.9)$$

$$2p(p + \gamma m)(B^k)_{i_1, \dots, i_{n-2p}}^m + (n - 2p + 1)(n - 2p + 2)(B^k)_{h, h, i_1, \dots, i_{n-2p}}^m = 0$$

$$\text{for } 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \quad , \quad (3.10)$$

$$2p(p + \gamma m + 1)(B^+)_{i_1, \dots, i_{n-2p-1}}^m + (n - 2p)(n - 2p + 1)(B^+)_{h, h, i_1, \dots, i_{n-2p-1}}^m = 0$$

$$\text{for } 1 \leq p \leq \left\lfloor \frac{n-1}{2} \right\rfloor \quad , \quad (3.11)$$

which follow from Eq.(2.15), are satisfied.

Note now that, as in the scalar case, the Cartesian components V^L obey Eq.(2.10)

$$\frac{\partial V^L}{\partial r} = 0 \quad . \quad (3.12)$$

The same condition for the covariant components in the spherical coordinates used in Eq.(3.3) reads

$$\frac{\partial V_\mu}{\partial r} = \frac{1}{r} V_\mu \quad , \quad (3.13)$$

because the matrix which defines the relation

$$V_\mu = \frac{\partial x^L}{\partial u^\mu} V_L$$

between two systems, scales as r . Eq.(3.13) does not change the action of the operator $-\nabla^K \nabla_K$ on $(V_\mu)_{n,m}$,

$$-\nabla^K \nabla_K (V_\mu)_{n,m} = \frac{1}{a^2} (n + \gamma m)(n + \gamma m + d - 1) (V_\mu)_{n,m} \quad . \quad (3.14)$$

Therefore according to Eqs.(3.3) and (3.13) we have the formula

$$-\nabla^\nu \nabla_\nu (V_\mu)_{n,m} = \frac{1}{a^2} [(n + \gamma m)(n + \gamma m + d - 1) - 1] (V_\mu)_{n,m} = \Lambda_{n,m}^T (V_\mu)_{n,m} \quad , \quad (3.15)$$

which gives the eigen-values $\Lambda_{n,m}^T$ in Eq.(1.5).

3.2 Degeneracies

We must now ensure that the vector field (3.6) is transverse and it is in the tangent space to S_β^d . These conditions read

$$x^K V_K = x^i V^i + x^+ V^- + x^- V^+ = 0 \quad , \quad (3.16)$$

$$\nabla_K V^K = \nabla_j V^j + \nabla_+ V^+ + \nabla_- V^- = 0 \quad , \quad (3.17)$$

where in the Cartesian basis $\nabla_K = \partial/\partial x^K$ and $\nabla_\pm = \partial/\partial x^\pm$. The first condition (3.16), after substitution of expression (3.6) takes the form

$$\begin{aligned} \sum_{p=1}^{\lfloor \frac{n+1}{2} \rfloor} (B^+)_{i_1, \dots, i_{n-2p+1}}^m x^{i_1} \dots x^{i_{n-2p+1}} (x^+ x^-)^p + \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} (B^k)_{i_1, \dots, i_{n-2p}}^m x^{i_1} \dots x^{i_{n-2p}} x^k (x^+ x^-)^p \\ + \sum_{p=0}^{\lfloor \frac{n+1}{2} \rfloor} (B^-)_{i_1, \dots, i_{n-2p+1}}^m x^{i_1} \dots x^{i_{n-2p+1}} (x^+ x^-)^p = 0 \end{aligned} \quad (3.18)$$

which is equivalent to a number of constraints

$$(n - 2p + 1) [(B^+)_{k, i_1, \dots, i_{n-2p}}^m + (B^-)_{k, i_1, \dots, i_{n-2p}}^m] + (B_{\{k\}}^m)_{i_1, \dots, i_{n-2p}} = 0 \quad , \quad (3.19)$$

where $1 \leq p \leq [(n - 1)/2]$ and $n > 1$, and

$$(B_{\{k\}}^m)_{i_1, \dots, i_n} + (n + 1)(B^-)_{k, i_1, \dots, i_n}^m = 0 \quad , \quad (3.20)$$

$$(B^+)^m + (B^-)^m = 0 \quad , \quad \text{for } n \text{ odd} \quad . \quad (3.21)$$

Here $(B_{\{k\}}^m)_{i_1, \dots, i_{n-2p}}$ is the expression which is completely symmetric in the all lower indexes, namely,

$$(B_{\{k\}}^m)_{i_1, \dots, i_{n-2p}} \equiv (B_k^m)_{i_1, \dots, i_{n-2p}} + \sum_{l=1}^{n-2p} (B_{i_l})_{i_1, \dots, i_{l-1}, k, i_{l+1}, \dots, i_{n-2p}}^m \quad , \quad (3.22)$$

$$(B_k^m)_{i_1, \dots, i_{n-2p}} = (B^k)_{i_1, \dots, i_{n-2p}}^m \quad .$$

Finally, if one uses (3.16), then transversality condition (3.17) results in equations

$$(\gamma m + p + 1)(B^+)_{i_1, \dots, i_{n-2p-1}}^m + (p + 1)(B^-)_{i_1, \dots, i_{n-2p-1}}^m + (n - 2p)(B_h)_{h, i_1, \dots, i_{n-2p-1}}^m = 0 \quad , \quad (3.23)$$

where $n \geq 1$ and $0 \leq p \leq [(n - 1)/2]$. As follows from restriction (3.7), in counting the degeneracies we should consider the cases $m \neq 0$ and $m = 0$ separately.

a) $m \neq 0$

For these values of m , Eqs.(3.9)–(3.11) enable one to express all the coefficients $(B^K)_{i_1, \dots, i_k}^m$ in terms of the coefficients $(B^K)_{i_1, \dots, i_{k+2}}^m$. Therefore the number of independent constants in (3.6) is completely determined by the number of independent quantities $(B^-)_{i_1, \dots, i_{n+1}}^m$, $(B^k)_{i_1, \dots, i_n}^m$, and $(B^+)_{i_1, \dots, i_{n-1}}^m$. However, $(B^+)_{i_1, \dots, i_{n-1}}^m$ are combinations of $(B^-)_{i_1, \dots, i_{n+1}}^m$ and $(B^k)_{i_1, \dots, i_n}^m$ because of Eq.(3.19) with $p = 1$. On the other hand, $(B^-)_{i_1, \dots, i_{n+1}}^m$ are given in terms of $(B^k)_{i_1, \dots, i_n}^m$ by Eq.(3.20). The explicit resolution of the constraints is represented in Appendix A by formulas (A.1)–(A.3). Let us note that in this way we get the coefficients which by the construction obey relations (3.9)–(3.11). What is more non-trivial is that also Eqs.(3.19) and (3.23) hold automatically on (A.1)–(A.3) for *any* number p . The same is true for Eq.(3.21). This can be checked by the direct substitution.

Thus the number of independent constants in (3.6) coincides with the number of components of the $(d-1)$ symmetric rank- n tensors $(B^k)_{i_1, \dots, i_n}^m$. The degeneracy

$$D_{n, m \neq 0}^T(d) = 2(d-1) \binom{n+d-2}{n} \quad (3.24)$$

is twice of this number because apart (3.6) one can construct (by interchanging x^+ and x^- in (3.6)) the analogous set of vectors with the same eigen-values.

b) $m = 0$

In this case, as follows from Eqs.(3.9)–(3.11) and (3.7), the number of independent constants in (3.6) is completely determined by the number of independent quantities $(B^-)_{i_1, \dots, i_{n-1}}^0$, $(B^k)_{i_1, \dots, i_n}^0$, and $(B^+)_{i_1, \dots, i_{n-1}}^0$. The tensors $(B^+)_{i_1, \dots, i_{n-1}}^0$ are combinations of $(B^-)_{i_1, \dots, i_{n-1}}^0$ and $(B^k)_{i_1, \dots, i_n}^0$ because of Eq.(3.19) with $p = 1$. However, $(B^-)_{i_1, \dots, i_{n-1}}^0$ -tensors are not related now to $(B^k)_{i_1, \dots, i_n}^0$ via (3.20), as it was in case $m \neq 0$, and they are determined in terms of $(B^-)_{i_1, \dots, i_{n-1}}^0$. Moreover from conditions (3.7) and (3.20) one gets the additional constraint

$$(B_{\{k\}})_{i_1, \dots, i_n}^0 = 0 \quad . \quad (3.25)$$

In summary, all the coefficients for $m = 0$ can be represented in terms of constants $(B^-)_{i_1, \dots, i_{n-1}}^0$ and $(B^k)_{i_1, \dots, i_n}^0$ with restriction (3.25). The explicit form of the solution is given in Appendix A by Eqs.(A.4)–(A.6). Again by the direct substitution of these solutions one can check that they satisfy Eq.(3.21) and constraints (3.19), (3.23) for any p . Thus the degeneracy of the eigen-value with $m = 0$

$$\begin{aligned} D_{n, 0}^T(d) &= \binom{n-1+d-2}{d-2} + (d-1) \binom{n+d-2}{d-2} - \binom{n+1+d-2}{d-2} \\ &= \frac{1}{n+1} \binom{d+n-3}{n-1} [d^2 + (n-4)d + (5-n)] \quad . \end{aligned} \quad (3.26)$$

is given by the number of symmetric tensors $(B^-)_{i_1, \dots, i_{n-1}}^0$ and $(B^k)_{i_1, \dots, i_n}^0$ minus the number of conditions (3.25).

The eigen-values (3.15) and degeneracies (3.24), (3.26) obtained on singular spheres S_β^d reproduce in the limit $\beta = 2\pi$ ($\gamma = 1$) the results of [28] for d -spheres. Indeed, if $\gamma = 1$ the eigen-values (3.15) can be expressed in terms of a single number $l = n + m$,

$$\Lambda_l^T(d) = \Lambda_{n,l-n}^T(d) = \frac{1}{a^2} (l(l+d-1) - 1) \quad , \quad l = 1, 2, \dots \quad , \quad (3.27)$$

and coincide with the eigen-values for the transverse vectors in [28]. Analogously, the total degeneracy is obtained from the equation

$$D_l^T(d) = D_{l,m=0}^T(d) + 2 \sum_{m=1}^l D_{l-m,0}^T(d) = \frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!} \quad , \quad (3.28)$$

with the help of summation formula (2.19) and it coincides with the result [28].

4 Heat-coefficients of vector Hodge-deRham operator

4.1 Relation to ζ -function

We now consider ζ -function and heat-coefficients for the vector Hodge-deRham operator $\Delta^{(1)}$ which is defined as

$$\Delta^{(1)} V^\nu = - \left(\nabla^\mu \nabla_\mu \delta_\rho^\nu - R_\rho^\nu \right) V^\rho \quad , \quad (4.1)$$

where $R_\rho^\nu = a^{-2}(d-1)\delta_\rho^\nu$ is the Ricci tensor computed in the regular domain of S_β^d . As we mentioned before the transversal and longitudinal parts of the vector field have to be studied separately. For the transversal components the ζ -function reads

$$\zeta^T(z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} D_{n,m}^T (\lambda_{n,m}^T)^{-z} + \sum_{n=1}^{\infty} D_{n,0}^T (\lambda_{n,0}^T)^{-z} \quad , \quad (4.2)$$

where the degeneracies are given by (3.24), (3.26), while, according to Eq.(4.1), the eigen-values $\lambda_{n,m}^T$ are

$$\lambda_{n,m}^T = \Lambda_{n,m}^T + a^{-2}(d-1) = \frac{1}{a^2} ((n+\gamma m)(n+\gamma m+d-1) + d-2) \quad . \quad (4.3)$$

The ζ -function is known to be the key object of the one-loop quantum-field computations on curved backgrounds [22]. In the present paper we use this function to study the coefficients in the Schwinger-DeWitt expansion of the heat kernel operator $K^T(s)$

$$K^T(s) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} D_{n,m}^T e^{-s\lambda_{n,m}^T} + \sum_{n=1}^{\infty} D_{n,0}^T e^{-s\lambda_{n,0}^T} \quad . \quad (4.4)$$

The quantities $\zeta^T(z)$ and $K^T(s)$ are related via the Mellin transformation²

$$\zeta^T(z) = \frac{1}{\Gamma(z)} \int_0^\infty ds \, s^{1-z} K^T(s) \quad , \quad (4.5)$$

²In the presence of zero modes, which is not our case, the relation (4.5) modifies.

where $\Gamma(z)$ is the gamma-function. It enables one to express the coefficients A_n in the asymptotic expansion of $K^T(s)$ at $s \rightarrow 0$

$$K^T(s) = \frac{1}{(4\pi s)^d} \sum_{n=0,1,2,\dots}^{\infty} A_n^T s^n \quad (4.6)$$

in terms of the residues of the ζ -function [30]

$$A_n^T = (4\pi)^{d/2} \Gamma\left(\frac{d}{2} - n\right) \text{Res} \left[\zeta^T\left(\frac{d}{2} - n\right) \right] , \quad (4.7)$$

for an arbitrary large dimension $d > 2n$, and as

$$A_{d/2}^T = (4\pi)^{d/2} \zeta^T(0) \quad (4.8)$$

if $d = 2n$. We will be interested in the coefficients A_1 and A_2 because they are important for studying the ultraviolet divergences and conformal anomalies in four dimensions.

The representation of the ζ -function which is convenient for using Eqs.(4.7) and (4.8) is found in Appendix B

$$\zeta^T(z) = \zeta_1(z) + \zeta_2(z) + \zeta_3(z) , \quad (4.9)$$

$$\begin{aligned} \zeta_1(z) &= \frac{2(d-1)}{\gamma(d-2)!} \sum_{k=0}^{\infty} C_k(z) \sigma^k \sum_{p=0}^{d-2} \sum_{l=0}^p \binom{p}{l} w_p \sum_{q=0}^{p-l} \binom{p-l}{q} q! \\ &\times (-1)^{p+l} \left(\frac{d-1}{2}\right)^{p-l-q} \frac{\Gamma(2z+2k-l-q-1)}{\Gamma(2z+2k-l)} \zeta_R\left(2z+2k-l-q-1, \frac{d-1}{2}\right) , \end{aligned} \quad (4.10)$$

$$\begin{aligned} \zeta_2(z) &= \frac{2(d-1)}{(d-2)!} \sum_{k=0}^{\infty} C_k(z) \sigma^k \sum_{r=0}^{\infty} \sum_{p=0}^{d-2} \sum_{l=0}^p \binom{p}{l} w_p \sum_{q=0}^{p-l} \binom{p-l}{q} (-1)^{p+l+q} \left(\frac{d-1}{2}\right)^{p-l-q} \\ &\times \frac{B_{r+q+1}}{r+q+1} \frac{\gamma^{r+q}}{r!} \frac{\Gamma(2z+2k+r-l)}{\Gamma(2z+2k-l)} \zeta_R\left(2z+2k+r-l, \frac{d-1}{2}\right) , \end{aligned} \quad (4.11)$$

$$\begin{aligned} \zeta_3(z) &= \frac{1}{(d-2)!} \sum_{k=0}^{\infty} C_k(z) \sigma^k \sum_{p=0}^{d-3} \sum_{l=0}^p \binom{p}{l} w'_p (-1)^{p+l} \left(\frac{d+1}{2}\right)^{p-l} \\ &\times \left[\frac{1}{2} (d-3)^2 \zeta_R\left(2z+2k-l, \frac{d+1}{2}\right) + (d-1) \zeta_R\left(2z+2k-l-1, \frac{d+1}{2}\right) \right] . \end{aligned} \quad (4.12)$$

In these expressions

$$\sigma = \frac{(d-3)^2}{4} , \quad (4.13)$$

$$C_k(z) = \frac{\Gamma(z+k)}{k!\Gamma(z)} \quad , \quad (4.14)$$

B_q are the Bernoulli numbers and quantities w_p, w'_p are defined as follows:

$$\sum_{p=0}^{d-2} n^p w_p(d) = \frac{(d+n-2)!}{n!} \quad , \quad (4.15)$$

$$\sum_{p=0}^{d-3} n^p w'_p(d) = \frac{(d+n-2)!}{(n+2)n!} \quad . \quad (4.16)$$

In the functions $\zeta_i(z)$ we have put for simplicity $a = 1$, because the dependence on the radius a factorizes in a overall multiplier a^{2z} , and thus it can be restored in the end of calculation. The expressions (4.10)–(4.12) hold when $z \rightarrow 0, -1, -2, \dots$. Although these formulas look complicated, they are very convenient for our purpose. Indeed, under more close examination one can see that for $\zeta_i(0)$ the series reduce to a finite sum, while in the limit $z \rightarrow -1, -2, \dots$ the functions $\zeta_i(z)$ have only a finite number of poles. In principle Eqs.(4.10)–(4.12) enable one to find the heat-kernel coefficients A_n with the help of Eq.(4.7) for arbitrary d and n .

In particular, with the help of Eqs.(B.9)–(B.14) and after tedious but simple computations one obtains

$$\frac{\bar{A}_1^T(d)}{(4\pi)^{d/2}} = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{12\gamma(d-3)!} \left[d(d-7) + (d-1)(\gamma^2 - 1) + 12(1-\gamma) \right] + \delta_{d,2} \quad , \quad (4.17)$$

$$\begin{aligned} \frac{\bar{A}_2^T(d)}{(4\pi)^{d/2}} &= \frac{\Gamma\left(\frac{d}{2}\right)}{360\gamma(d-1)!} \left(5d^5 - 77d^4 + 325d^3 - 369d^2 + 246d + (d-1)(d-3) \right. \\ &\times \left. \left[2(d-1)(1-\gamma^4) + 10(d-1)(d-4)(\gamma^2 - 1) + 120(d-2)(1-\gamma) \right] \right) + \delta_{d,4} \quad , \quad (4.18) \end{aligned}$$

where $\delta_{d,k}$ is the Kronecker symbol. The bar on the above coefficients stresses that they are computed on the singular spaces. The origin of the last terms in the right hand side (r.h.s.) of Eqs.(4.17) and (4.18) is explained by the different expressions of the heat coefficients when $d > 2n$ and $d = 2n$, see Eqs. (4.7), (4.8). Note that the results (4.7), (4.8) are obtained for $\beta \leq 2\pi$, and they can be analytically continued to $\beta > 2\pi$.

4.2 A_1 -coefficient

Consider now the complete coefficients $A_n^{(1)}$ of the vector Hodge-deRham operator. On S_β^d they are the sum

$$\bar{A}_n^{(1)} = \bar{A}_n^T + \bar{A}_n^L = \bar{A}_n^T + \bar{A}_n^{(0)} - (4\pi)^{d/2} \delta_{d,2n} \quad (4.19)$$

of the heat coefficients \bar{A}_n^T and \bar{A}_n^L corresponding to the transversal and longitudinal components, respectively. On the other hand, the longitudinal coefficient \bar{A}_n^L is related to

the coefficient $\bar{A}_n^{(0)}$ of the scalar Laplacian from which one has to remove the contribution of the single zero mode when $n = d/2$, see Eq.(3.3).

For the scalar Laplacian on a closed space \mathcal{M}_β with conical singularities having deficit angle $2\pi - \beta$ and forming the hypersurface Σ

$$\bar{A}_1^{(0)} = \frac{1}{6} \int_{\mathcal{M}_\beta - \Sigma} R + A_{\beta,1}^{(0)} \quad , \quad (4.20)$$

$$A_{\beta,1}^{(0)} = \frac{\pi}{3\gamma} (\gamma^2 - 1) \int_{\Sigma} \quad , \quad (4.21)$$

where, as before, $\gamma = 2\pi/\beta$. The general expression of the vector coefficient is [25]

$$\bar{A}_1^{(1)} = \frac{d-6}{6} \int_{\mathcal{M}_\beta - \Sigma} R + A_{\beta,1}^{(1)} \quad , \quad (4.22)$$

$$A_{\beta,1}^{(1)} = d A_{\beta,1}^{(0)} + \frac{4\pi}{\gamma} (1 - \gamma) \int_{\Sigma} \quad . \quad (4.23)$$

Note that $A_{\beta,1}^{(0)}$, $A_{\beta,1}^{(1)}$ are the corrections to the heat coefficients due to conical singularities. These corrections are proportional to the volume of the surface Σ , which is S^{d-2} in the case under consideration. One can check³ by substitution of (4.17) and (4.20) into Eq.(4.19) with $n = 1$, that it reproduces the exact formula (4.22) for d -spheres S_β^d . Therefore, the direct mode-by-mode computation of the first vector coefficient confirms the general analysis of [25].

4.3 A_2 -coefficient

The form of this coefficient in vector case is not yet known on singular spaces. On the smooth closed manifolds (see for instance [15, 31])

$$A_2^{(0)} = \frac{1}{180} \int_{\mathcal{M}} \left(R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - R_{\mu\nu} R^{\mu\nu} + \frac{5}{2} R^2 \right) \quad , \quad (4.24)$$

$$A_2^{(1)} = d A_2^{(0)} + \int_{\mathcal{M}} \left(-\frac{1}{12} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + \frac{1}{2} R_{\mu\nu} R^{\mu\nu} - \frac{1}{6} R^2 \right) \quad , \quad (4.25)$$

where $R_{\mu\nu\lambda\rho}$ and $R_{\mu\nu}$ are the components of the Riemann and Ricci tensors respectively. First of all, with the help of Eqs.(4.19),(4.24) and (4.25) one can check that A_2^T , Eq.(4.18), has the correct expression on S^d when $\gamma = 1$.

Another interesting exercise is the comparison of (4.18) at the small deficit angle ($\gamma \simeq 1$) with the behaviour of $A_2^{(1)}$ on manifolds with the "blunted" singularities [25]. Indeed, one can consider the coefficients (4.24) and (4.25) on a sequence of smooth manifolds $\tilde{\mathcal{M}}_\beta$ which converges to a space \mathcal{M}_β with conical singularities. We refer to $\tilde{\mathcal{M}}_\beta$ as to manifolds with the blunted singularities. One can prove [7] that the following formulas:

$$\int_{\tilde{\mathcal{M}}_\beta} R^2 \simeq \int_{\mathcal{M}_\beta - \Sigma} R^2 + 8\pi(\gamma - 1) \int_{\Sigma} R \quad , \quad (4.26)$$

³ It is worth reminding that the volume of S_β^d is $(4\pi)^{d/2} \Gamma(d/2) / (\gamma(d-1)!)$.

$$\int_{\tilde{\mathcal{M}}_\beta} R_{\mu\nu} R^{\mu\nu} \simeq \int_{\mathcal{M}_\beta - \Sigma} R_{\mu\nu} R^{\mu\nu} + 4\pi(\gamma - 1) \int_{\Sigma} R_{ii} \quad , \quad (4.27)$$

$$\int_{\tilde{\mathcal{M}}_\beta} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \simeq \int_{\mathcal{M}_\beta - \Sigma} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + 8\pi(\gamma - 1) \int_{\Sigma} R_{ijij} \quad , \quad (4.28)$$

are valid up to the leading order in $(\gamma - 1)$ when $\tilde{\mathcal{M}}_\beta \rightarrow \mathcal{M}_\beta$. Here $R_{ii} = R_{\mu\nu} n_i^\mu n_i^\nu$ and $R_{ijij} = R_{\mu\nu\lambda\rho} n_i^\mu n_j^\nu n_i^\lambda n_j^\rho$ are the components of the Ricci and Riemann tensors computed in the regular domain near Σ and normal to this surface. The quantities n_i^μ ($i = 1, 2$) are two unit normal vectors. On the d -spheres with unit radius

$$R = d(d - 1) \quad , \quad (4.29)$$

$$R_{ii} = 2(d - 1) \quad , \quad (4.30)$$

$$R_{ijij} = 2 \quad . \quad (4.31)$$

Thus, by taking into account Eqs.(4.19), (4.24)–(4.31) one can verify that for small values of $|1 - \gamma|$ the coefficients (4.18) can be approximated by the standard coefficients evaluated on the spheres \tilde{S}_β^d with the blunted conical singularities

$$\bar{A}_2^T[S_\beta^d] = A_2^{(1)}[\tilde{S}_\beta^d] - A_2^{(0)}[\tilde{S}_\beta^d] + (4\pi)^{d/2} \delta_{d,2n} + O((1 - \gamma)^2) \quad . \quad (4.32)$$

As it was shown in [23], the same property holds in general for the coefficients of the scalar Laplacians (without curvature couplings). Hence, one can also conclude that

$$\bar{A}_2^{(1)}[S_\beta^d] = A_2^{(1)}[\tilde{S}_\beta^d] + O((1 - \gamma)^2) \quad . \quad (4.33)$$

Note that Eq.(4.33) is a nice property of the vector Hodge-deRham operator $\Delta^{(1)}$ only. Arbitrary vector Laplacians, like $-\nabla_\mu \nabla^\mu$ for example, which differ from $\Delta^{(1)}$ by terms depending on curvature, do not have this property.

Let us discuss now the form of $\bar{A}_2^{(1)}$ on arbitrary manifolds \mathcal{M}_β with conical singularities. Expression (4.25) for the standard coefficient suggests that this form can be similar to that of the first vector coefficient, Eqs.(4.22),(4.23),

$$\bar{A}_2^{(1)} = A_2^{(1)} + A_{\beta,2}^{(1)} \quad , \quad (4.34)$$

$$A_{\beta,2}^{(1)} = d A_{\beta,2}^{(0)} + Q_{\beta,2} \quad . \quad (4.35)$$

Here $A_{\beta,2}^{(0)}$ is the correction to the scalar coefficient $A_2^{(0)}$ (4.24) due to conical singularities [19]

$$A_{\beta,2}^{(0)} = \frac{\pi}{90\gamma} \int_{\Sigma} \left[(1 - \gamma^4) \left(\frac{1}{2} R_{ii} - R_{ijij} \right) - 5(1 - \gamma^2) R \right] \quad . \quad (4.36)$$

$Q_{\beta,2}$ is a functional similar to $A_{\beta,2}^{(0)}$

$$Q_{\beta,2} = \frac{\pi}{\gamma} \int_{\Sigma} \left[(1 - \gamma^4) (a_1 R + a_2 R_{ii} + a_3 R_{ijij}) \right]$$

$$+(1-\gamma^2)(b_1R+b_2R_{ii}+b_3R_{ijij})+(1-\gamma)(c_1R+c_2R_{ii}+c_3R_{ijij}) \quad , \quad (4.37)$$

where a_k , b_k and c_k are numerical coefficients. This structure of $Q_{\beta,2}$ results from considerations similar to those of Dowker [20]. Let the dimensionality of the parameter s in the Schwinger-DeWitt expansion be L^2 , where L is a length. Then, as it follows from (4.6), the dimensionality of $Q_{\beta,2}$ is L^{d-4} . On the other hand, the integral \int_{Σ} scales as L^{d-2} and, consequently the integrand in (4.37) has the dimension L^{-2} . According to the Gauss-Codacci equations, see Ref.[19], one can find only three independent invariants with such dimension: R , R_{ii} and R_{ijij} , which describe internal and external geometry of Σ .

Fortunately, all the coefficients in (4.37) can be fixed if one makes use of Eq.(4.19) and expression (4.18) for the transversal vectors on S_{β}^d . After simple computations we get Eq. (1.8)

$$A_{\beta,2}^{(1)} = d A_{\beta,2}^{(0)} + \frac{\pi}{3\gamma} \int_{\Sigma} \left[(1-\gamma^2)(R - R_{ii}) + 2(1-\gamma)(R - 2R_{ii} + R_{ijij}) \right] \quad . \quad (4.38)$$

It would be interesting to confirm this result by the direct computations analogous to those of [19],[21] without assumptions (4.36),(4.37).

5 Renormalization of the gauge effective action

Let us discuss now renormalization in quantum field theory on singular spaces \mathcal{M}_{β} . As it was mentioned in the Section 1 the geometrical structure of the divergences W_{div} in quantum theory on curved backgrounds is determined by the coefficients A_0 , A_1 and A_2 , see (1.3). The effective action W for the gauge fields quantized with the gauge condition $\nabla^{\mu}V_{\mu} = 0$ is given in terms of the determinant of the Hodge-deRham operator by Eq. (1.1), while the contribution of ghosts has the form of the determinant of the scalar Laplacian. It follows from the results of Ref.[25] and Eq.(4.38) for $A_{\beta,2}^{(1)}$, that in the leading order in deficit angle $(2\pi - \beta)$

$$\bar{A}_n^{(1)}[\mathcal{M}_{\beta}] \simeq A_n^{(1)}[\tilde{\mathcal{M}}_{\beta}] + O((2\pi - \beta)^2) \quad , \quad n = 1, 2 \quad . \quad (5.1)$$

It means that one can approximate the coefficients on the singular spaces \mathcal{M}_{β} by their expressions on the corresponding smooth manifolds $\tilde{\mathcal{M}}_{\beta}$. As for A_0 , it does not depend on conical singularities. Then Eq.(1.1) says that the same property is valid for the divergent part of the gauge action

$$W_{\text{div}}[\mathcal{M}_{\beta}] \simeq W_{\text{div}}[\tilde{\mathcal{M}}_{\beta}] + O((2\pi - \beta)^2) \quad . \quad (5.2)$$

Hence, in this order the one-loop divergences on \mathcal{M}_{β} are completely removed by the *standard* renormalization procedure [22] used on the smooth backgrounds, i.e. by the renormalization of the Newton constant and couplings by the terms R^2 , $R_{\mu\nu}^2$ and $R_{\mu\nu\lambda\rho}^2$ in the bare gravitational Lagrangian. The effective action of the scalar fields without curvature couplings has the analogous property [23] and, hence, so does the ghost action.

This conclusion is important for black hole thermodynamics in the off-shell formulation [6]–[14]. In this case \mathcal{M}_β appear as the Euclidean section of the corresponding Lorentzian space-times, Σ represents the Euclidean horizon and the imaginary time period β is associated with the inverse temperature of the system. The regularity condition $\beta = 2\pi$ corresponds to the Hawking temperature. For off-shell computations of the black hole entropy S it is sufficient to use the decomposition of the effective action W near $\beta = 2\pi$ up to the terms $\sim (2\pi - \beta)$. Our results mean that in case of gauge fields the ultraviolet divergences in S are removed under the standard renormalization of coupling constants in the bare gravitational action, analogously to scalar fields [23]. It also indicates the agreement between on-shell and off-shell computations of the thermodynamical quantities for black holes [5].

6 Summary and perspectives

The main result of this paper is the exact computation of the spectrum (1.5)–(1.7) of the vector Laplacian on d -spheres S_β^d with the conical singularities. This is the generalization of the results obtained by Rubin and Ordonez [28] on usual spheres S^d . The spectrum can be used now for the computation of the gauge one-loop effective action in terms of the ζ -function (4.9)–(4.12). This will be a development of works by Allen et al. [32]–[34] investigated the effective potential in gauge models in the de Sitter space.

In the present paper we used the spectrum (1.5)–(1.7) to compute the coefficients of the heat kernel expansion on S_β^d and to study their properties. Computation of $\bar{A}_1^{(1)}$ confirms the results of [24] and [25] found by different methods. We have also calculated the second coefficient $\bar{A}_2^{(1)}$ on S_β^d and suggested its generalization (4.38) to arbitrary manifolds \mathcal{M}_β with conical singularities. This result is both new and important because $\bar{A}_2^{(1)}$ is related to the conformal anomaly.

Finally we have shown that the vector Hodge-deRham operator on \mathcal{M}_β has a number of properties analogous to those of the scalar Laplacian $-\nabla^\mu \nabla_\mu$. The heat coefficients for small deficit angles can be approximated by the coefficients on the corresponding spaces with the blunted singularities. The immediate consequence of these properties is the validity for gauge fields of a renormalization theorem concerning the black hole entropy proven in [23] for scalar fields.

The method employed here to find the spectrum (1.5)–(1.7) is straightforward but it is quite simple. It would be interesting to use it to get the spectra of the Laplacians of rank 2 symmetric tensors on S_β^d and to continue investigation of these operators on singular spaces started in [25]. Studying this problem is in progress.

Acknowledgements One of the authors (D.V.F) is very grateful to University of Naples for hospitality and to INFN for the support during his visit to Naples. This work was also supported in part by the Natural Sciences and Engineering Research Council of Canada.

A Explicit solution of constraints

We present here the explicit solution of constraints (3.9)–(3.11) and (3.19)–(3.23) given in terms of the coefficients $(B^k)_{i_1, \dots, i_n}^m$ with $m = 0, 1, 2, \dots$, and $(B^-)_{i_1, \dots, i_{n-1}}^0$. The method of its construction is described in Section 3.

If $m \neq 0$ the solution is

$$(B^-)_{i_1, \dots, i_{n-2p+1}}^m = \frac{(-1)^{p+1}}{2^p p!} \frac{n!}{(n-2p+1)!} \frac{\Gamma(\gamma m)}{\Gamma(\gamma m + p)} (B_{\{h_1\}_{h_1, \dots, h_p, h_p, i_1, \dots, i_{n-2p+1}}})^m \quad \text{for } 0 \leq p \leq \left\lfloor \frac{n+1}{2} \right\rfloor, \quad (\text{A.1})$$

$$(B^k)_{i_1, \dots, i_{n-2p}}^m = \frac{(-1)^p}{2^p p!} \frac{n!}{(n-2p)!} \frac{\Gamma(\gamma m + 1)}{\Gamma(\gamma m + p + 1)!} (B^k)_{h_1, h_1, \dots, h_p, h_p, i_1, \dots, i_{n-2p}}^m \quad \text{for } 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor, \quad (\text{A.2})$$

$$(B^+)_{i_1, \dots, i_{n-2p-1}}^m = \frac{(-1)^p}{2^{p+1} p!} \frac{n!}{(n-2p-1)!} \frac{\Gamma(\gamma m + 2)}{\Gamma(\gamma m + p + 2)} \times \left[\frac{1}{\gamma m + 1} (B_{\{h_1\}_{h_1, \dots, h_p, h_p, i_1, \dots, i_{n-2p-1}}})^m - \frac{1}{\gamma m} (B_{\{h_1\}_{h_1, \dots, h_{p+1}, h_{p+1}, i_1, \dots, i_{n-2p-1}}})^m \right] \quad \text{for } 0 \leq p \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (\text{A.3})$$

For $m = 0$ one can find

$$(B^-)_{i_1, \dots, i_{n-2p+1}}^0 = \frac{(-1)^{p+1}}{2^{(p-1)p!} (p-1)!} \frac{(n-1)!}{(n-2p+1)!} (B^-)_{h_1, h_1, \dots, h_{p-1}, h_{p-1}, i_1, \dots, i_{n-2p+1}}^0 \quad \text{for } 2 \leq p \leq \left\lfloor \frac{n+1}{2} \right\rfloor, \quad (\text{A.4})$$

$$(B^+)_{i_1, \dots, i_{n-2p-1}}^0 = \frac{(-1)^{p+1}}{2^p p! (p+1)!} \frac{(n-1)!}{(n-2p-1)!} \left[(B^-)_{h_1, h_1, \dots, h_p, h_p, i_1, \dots, i_{n-2p-1}}^0 + n (B_{h_1}_{h_1, \dots, h_{p+1}, h_{p+1}, i_1, \dots, i_{n-2p-1}})^0 \right] \quad \text{for } 0 \leq p \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (\text{A.5})$$

$$(B^k)_{i_1, \dots, i_{n-2p}}^0 = \frac{(-1)^p}{2^p (p!)^2} \frac{n!}{(n-2p)!} (B^k)_{h_1, h_1, \dots, h_p, h_p, i_1, \dots, i_{n-2p}}^0 \quad \text{for } 0 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (\text{A.6})$$

One can verify by a straightforward substitution that the solutions (A.1)–(A.3) and (A.4)–(A.6) satisfy all constraints (3.19)–(3.21), (3.23) and relations (3.9)–(3.11).

B ζ -function

Here we give the details how to derive representation (4.9)–(4.12) for the ζ -function. We follow the method developed in [27] for the scalar ζ -function.

First, we use the decomposition

$$(\lambda_{n,m}^T)^{-z} = \sum_{k=0}^{\infty} C_k(z) \sigma^k \left(n + \gamma m + \frac{d-1}{2} \right)^{-2z-2k}, \quad (\text{B.1})$$

where σ and $C_k(z)$ are defined by Eqs.(4.13),(4.14) respectively. Then, by taking into account definitions (4.15) and (4.16) we obtain

$$\begin{aligned}\zeta_1(z) + \zeta_2(z) &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} D_{n,m}^T (\lambda_{n,m}^T)^{-z} \\ &= \frac{2(d-1)}{(d-2)!} \sum_{k=0}^{\infty} C_k(z) \sigma^k \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(n + \gamma m + \frac{d-1}{2} \right)^{-2z-2k} \sum_{p=0}^{d-2} n^p w_p \quad , \quad (\text{B.2})\end{aligned}$$

$$\begin{aligned}\zeta_3(z) &= \sum_{n=1}^{\infty} D_{n,0}^T (\lambda_{n,0}^T)^{-z} \\ &= \frac{1}{(d-2)!} \sum_{k=0}^{\infty} C_k(z) \sigma^k \sum_{n=0}^{\infty} \left(n + \frac{d+1}{2} \right)^{-2z-2k} \left[d^2 + d(n-3) + 4 - n \right] \sum_{p=0}^{d-3} n^p w'_p \quad , \quad (\text{B.3})\end{aligned}$$

where in the r.h.s. of (B.3) the summation index n was replaced with $n+1$. Let us focus now on the computation of $\zeta_1(z) + \zeta_2(z)$, because the computation of $\zeta_3(z)$ is analogous. One can use in (B.2) the decomposition

$$n^p = \sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \left(n + \gamma m + \frac{d-1}{2} \right)^l \left(\gamma m + \frac{d-1}{2} \right)^{p-l} \quad (\text{B.4})$$

to get

$$\begin{aligned}\zeta_1(z) + \zeta_2(z) &= \frac{2(d-1)}{(d-2)!} \sum_{k=0}^{\infty} C_k(z) \sigma^k \\ &\times \sum_{p=0}^{d-2} \sum_{l=0}^p (-1)^{p+l} \binom{p}{l} w_p \sum_{m=1}^{\infty} \zeta_R \left(2z + 2k - l, \gamma m + \frac{d-1}{2} \right) \left(\gamma m + \frac{d-1}{2} \right)^{p-l} \quad , \quad (\text{B.5})\end{aligned}$$

where

$$\zeta_R(z, a) = \sum_{n=0}^{\infty} (n+a)^{-z} \quad , \quad a \neq 0, -1, -2, \dots$$

is the Riemann ζ -function. The integral representation of $\zeta_R(z, a)$

$$\zeta_R(z, a) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{y^{z-1} e^{-ay}}{1 - e^{-y}} dy \quad (\text{B.6})$$

enables one to sum up over m in Eq.(B.5)

$$\begin{aligned}&\sum_{m=1}^{\infty} \zeta_R \left(2z + 2k - l, \gamma m + \frac{d-1}{2} \right) \left(\gamma m + \frac{d-1}{2} \right)^{p-l} \\ &= \frac{1}{\Gamma(2z + 2k - 2l)} \int_0^{\infty} dy \frac{y^{2z+2k-l-1}}{1 - e^{-y}} (-1)^{p-l} \left(\frac{d}{dy} \right)^{p-l} \left[\frac{\exp \left(-\frac{d-1}{2} y \right)}{e^{\gamma y} - 1} \right] \quad . \quad (\text{B.7})\end{aligned}$$

Then one can use in (B.7) the definition of the Bernoulli numbers B_n

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \quad .$$

This gives

$$\begin{aligned}
& \sum_{m=1}^{\infty} \zeta_R \left(2z + 2k - l, \gamma m + \frac{d-1}{2} \right) \left(\gamma m + \frac{d-1}{2} \right)^{p-l} \\
&= \frac{1}{\gamma} \sum_{q=0}^{p-l} \binom{p-l}{q} \left(\frac{d-1}{2} \right)^{p-l-q} q! \frac{\Gamma(2z + 2k - l - q - 1)}{\Gamma(2z + 2k - l)} \zeta_R \left(2z + 2k - l - q - 1, \frac{d-1}{2} \right) \\
&+ \sum_{r=0}^{\infty} \sum_{q=0}^{p-l} \binom{p-l}{q} \left(\frac{d-1}{2} \right)^{p-l-q} (-1)^q \frac{B_{r+q+1}}{r+q+1} \frac{\gamma^{r+q}}{r!} \\
&\times \frac{\Gamma(2z + 2k + r - l)}{\Gamma(2z + 2k - l)} \zeta_R \left(2z + 2k + r - l, \frac{d-1}{2} \right) \quad . \quad (B.8)
\end{aligned}$$

Now the substitution of (B.8) into (B.5) gives the expressions for $\zeta_1(z)$ and $\zeta_2(z)$. The first term in r.h.s. of (B.8) results in expression (4.10), while the second term in r.h.s. of (B.8) corresponds to formula (4.11).

Finally we give explicitly several numbers w_p , see Eq.(4.15), which are used to get Eqs.(4.17) and (4.18)

$$w_{d-2} = 1 \quad , \quad (B.9)$$

$$w_{d-3} = \frac{1}{2}(d-1)(d-2) \quad , \quad (B.10)$$

$$w_{d-4} = \frac{1}{24}(d-1)(d-2)(d-3)(3d-4) \quad , \quad (B.11)$$

$$w_{d-5} = \frac{1}{48}(d-1)^2(d-2)^2(d-3)(d-4) \quad , \quad (B.12)$$

$$w_{d-6} = \frac{1}{5760}(d-1)(d-2)(d-3)(d-4)(d-5)(15d^3 - 75d^2 + 110d - 48) \quad . \quad (B.13)$$

Regarding the numbers w'_p in formula (4.16), they can be found from w_p with the help of relations

$$w'_{d-3} = 1 \quad , \quad w'_{p-1} = w_p - 2w'_p \quad , \quad (B.14)$$

for $1 \leq p \leq d-4$.

References

- [1] A.A. Vilenkin, Phys. Rep. **121**, N5 (1985) 263.
- [2] G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15** (1977) 2752.
- [3] V.P. Frolov, Phys. Rev. Lett. **74** (1995) 3319.
- [4] S. Carlip and C. Teitelboim, Class. Quantum Grav. **12** (1995) 1699, C. Teitelboim, Phys. Rev. **D53** (1996) 2870.
- [5] V.P. Frolov, D.V. Fursaev and A.I. Zelnikov, Phys. Rev. **D54** (1996) 2711, hep-th/9512184.
- [6] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **72** (1994) 957.
- [7] D.V. Fursaev and S.N. Solodukhin, Phys. Rev. **D52** (1995) 2133.
- [8] L. Susskind and J. Uglum, Phys. Rev. **D50** (1994) 2700.
- [9] C. Callan and F. Wilczek, Phys. Lett. **B333** (1994) 55.
- [10] S.N. Solodukhin, Phys. Rev. **D51** (1994) 609, ibid **D51** (1995) 618, D.V. Fursaev, Mod. Phys. Lett. **A10** (1995) 649.
- [11] D.V. Fursaev, Phys. Rev. **D51** (1995) R5352.
- [12] V.P. Frolov, W. Israel, and S.N. Solodukhin, *On One-loop Quantum Corrections to the Thermodynamics of Charged Black Holes*, ALBERTA-THY-06-96, hep-th/9602105.
- [13] R.B. Mann and S.N. Solodukhin, *Conical Geometry and Quantum Entropy of a Charged Kerr Black Hole*, preprint WATPHYS-TH-96-04, hep-th/9604118.
- [14] R.B. Mann and S.N. Solodukhin, *Quantum scalar field on three-dimensional (BTZ) black hole instanton: heat kernel, effective action and thermodynamics*, preprint WATPHYS-TH-96-11, hep-th/9609085.
- [15] B.C. DeWitt, Phys. Rep. **19C** (1975) 297.
- [16] H. Donnelly, Math. Ann. **224** (1976) 161.
- [17] G. Cognola, K. Kirsten and L. Vanzo, Phys. Rev. **D49** (1994) 1029.
- [18] D.V. Fursaev, Class. Quantum Grav. **11** (1994) 1431.
- [19] D.V. Fursaev, Phys. Lett. **B334** (1994) 53.
- [20] J.S. Dowker, Class. Quantum Grav. **11** (1994) L137.

- [21] J.S. Dowker, Phys. Rev. **D50** (1994) 6369.
- [22] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge 1982.
- [23] D.V. Fursaev and S.N. Solodukhin, Phys. Lett. **B365** (1996) 51.
- [24] D. Kabat, Nucl. Phys. **453** (1995) 281.
- [25] D.V. Fursaev and G. Miele, *Spins, Cones and Heat kernels*, preprint ALBERTA-THY 17-96, DSF-T-23/96, hep-th/9605153.
- [26] F. Larsen and F. Wilczek, Nucl. Phys. **B458** (1996) 249.
- [27] D.V. Fursaev and G. Miele, Phys. Rev. **D49** (1994) 987.
- [28] M.A. Rubin and C.R. Ordonez, J. Math. Phys. **25** (1984) 2888.
- [29] E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **12** (1963) 185.
- [30] R. Camporesi, Phys. Rep. **196** (1990) 1.
- [31] A.O. Barvinsky and G.A. Vilkovisky, Phys. Rep. **119** (1985) 1.
- [32] B. Allen, Nucl. Phys. **226** (1983) 228; Ann. Phys. **161** (1985) 152.
- [33] F. Buccella, G. Esposito and G. Miele, Class. Quantum Grav. **9** (1992) 1499.
- [34] G. Esposito, G. Miele and L. Rosa, Class. Quantum Grav. **11** (1994) 2031.